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STEADY NONLINEAR HARMONIC OSCILLATIONS OF DISCRETE ELASTIC STRUCTURES

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Abstract—An algorithm is derived which can be used for the calculation of coefficients in the amplitude frequency equations for nonlinear harmonic oscillations of elastic structures using, e.g. the finite element method.

INTRODUCTION

A basic problem in analysis of steady nonlinear harmonic oscillations of elastic structures is to determine the dependence of the frequency of oscillation on the amplitude of vibration. In what follows we present an algorithm which can be used for such calculations in conjunction with the finite element method. Although our procedure is derived from a simple application of an asymptotic technique[1, 2], it appears to be of some interest mainly because a number of finite element-based solutions of nonlinear oscillations problems that have appeared in recent literature are based on a method which, as we show in the following, leads to incorrect results even for a single degree of freedom system and, therefore, does not seem to have general validity.

NONLINEAR HARMONIC OSCILLATIONS OF CONSERVATIVE STRUCTURAL SYSTEMS

Our purpose in this section is to present a perturbation procedure for analysis of steady, forced harmonic oscillations of conservative structural systems with damping. Thus, we consider the equations of motion

$$\mathbf{M}\mathbf{q}_{tt} + \mathbf{C}\mathbf{q}_t + \mathbf{K}\mathbf{q} + \mathbf{N}(\mathbf{q}) = \mathbf{f}(e^{i\omega t} + e^{-i\omega t}), \tag{1a}$$

$$\mathbf{N}(\mathbf{q}) \equiv \frac{1}{2}\mathbf{N1}(\mathbf{q})\mathbf{q} + \frac{1}{3}\mathbf{N2}(\mathbf{q})\mathbf{q}, \tag{1b}$$

$$\mathbf{q}(0) = \mathbf{q}(2\pi/\omega);\ \mathbf{q}_t(0) = \mathbf{q}_t(2\pi/\omega), \tag{1c}$$

where **K**, **M**, and **C** are symmetric matrices with constant elements, denoting the structural stiffness, mass and damping, respectively, and ω is the frequency of the forcing function. The matrices N1 and N2 in (1b) are symmetric and their elements are linear and quadratic functions, respectively, of the components of **q**, the vector denoting the degrees of freedom of the structure. Equations of the form (1) arise in analysis of thin elastic structures under the small strains, moderate rotations approximation and can be formulated following the method given, for example, in [3, 4].

To solve (1) under the assumption that magnitudes of damping and the forcing function are small, we explicitly introduce a small parameter μ given by

$$\mathbf{C} = \mu\mathbf{C}', \mathbf{f} = \mu\mathbf{f}', \tag{2}$$

where $\mathbf{C}'$ and $\mathbf{f}'$ are of order unity. We further introduce nondimensional time according to

$$t' = \omega t. \tag{3}$$

Substitution of (2), (3) into (1) furnishes, with dot denoting derivative with respect to nondimensional time,

$$\lambda\mathbf{M}\ddot{\mathbf{q}} + \mathbf{K}\mathbf{q} + \mathbf{N}(\mathbf{q}) + \mu[\sqrt{(\lambda)}\mathbf{C}\dot{\mathbf{q}} - \mathbf{f}(e^{it} + e^{-it})] = 0, \tag{4a}$$

$$\mathbf{q}(0) = \mathbf{q}(2\pi);\ \dot{\mathbf{q}}(0) = \dot{\mathbf{q}}(2\pi), \tag{4b}$$

where we have dropped the primes from $\mathbf{C}'$, $\mathbf{f}'$ and t' and have introduced the definition

$$\lambda = \omega^2. \tag{5}$$

Equation (4) is now in the form of a set of two-point (nonlinear) boundary value problems with λ as a parameter. When $\mu = 0$, this set has the trivial solution for all values of λ. Condition for the existence of bifurcation points on the trivial solution branch yields the equations for the natural frequencies and free vibration modes of the structure; to wit,

$$\lambda_0 M\ddot{\mathbf{u}} + \mathbf{K}\mathbf{u} = 0 \tag{6a}$$

$$\mathbf{u}(0) = \mathbf{u}(2\pi);\ \dot{\mathbf{u}}(0) = \dot{\mathbf{u}}(2\pi). \tag{6b}$$

Equation (6) has solutions

$$\mathbf{u} = \mathbf{y}e^{it},\ \bar{\mathbf{u}} = \mathbf{y}e^{-it} \tag{7}$$

where $\mathbf{y}$ is the free vibration mode satisfying the linear eigenvalue problem

$$[-\lambda_0\mathbf{M} + \mathbf{K}]\mathbf{y} = 0\ \ \mathbf{y}^T\mathbf{y} = 1. \tag{8}$$

For the purpose of this note we assume that the free vibration mode excited by the forcing function is associated with a natural frequency such that (8) has only one linearly independent solution. Therefore, in the space of all 2π-periodic vector functions, i.e., among the vector functions of time that satisfy (4b), the differential operator

$$\mathbf{B} \equiv \left[\lambda_0\mathbf{M}\frac{d^2}{dt^2} + \mathbf{K}\right] \tag{9}$$

has a two dimensional null-space given by $\mathbf{u}$ and $\bar{\mathbf{u}}$.

We first note that any 2π-periodic vector function has the *unique* decomposition

$$\mathbf{x} = (\alpha\, e^{it} + \bar{\alpha}\, e^{-it})\mathbf{y} + \mathbf{w}. \tag{10}$$

In (10), α, $\bar{\alpha}$ and $\mathbf{w}$ are defined by using the operators Q_N, $\bar{Q}_N$ and $Q_{\mathbf{R}}$, thus,

$$\alpha = Q_N\mathbf{x} \equiv \frac{1}{2\pi}\int_0^{2\pi} e^{-it}\,\mathbf{y}^T\mathbf{x}\,dt \tag{11a}$$

$$\bar{\alpha} = \bar{Q}_N\mathbf{x} \equiv \frac{1}{2\pi}\int_0^{2\pi} e^{it}\,\mathbf{y}^T\mathbf{x}\,dt \tag{11b}$$

$$\mathbf{w} = Q_{\mathbf{R}}\mathbf{x} \equiv (\mathbf{I} - \mathbf{u}Q_N - \bar{\mathbf{u}}\bar{Q}_N)\mathbf{x} \tag{12}$$

where $\mathbf{I}$ is the identity matrix. Here w belongs to the range of the operator $\mathbf{B}$, and the range is orthogonal to the null-space spanned by u and $\bar{u}$, with respect to the scalar product

$$\langle \mathbf{w}_1, \mathbf{w}_2\rangle \equiv \frac{1}{2\pi}\int_0^{2\pi} \bar{\mathbf{w}}_1{}^T\mathbf{w}_2\,dt \tag{13}$$

where $\mathbf{w}_1$ and $\mathbf{w}_2$ are any 2π-periodic vector functions and overbar denotes a complex conjugate. As a result of the orthogonality of the range of $\mathbf{B}$ to the null space of the latter, any 2π-periodic function vanishes if and only if its components in the two spaces vanish, or, equivalently,

$$\mathbf{x} = 0 \longleftrightarrow Q_N\bar{\mathbf{x}} = \bar{Q}_N\mathbf{x} = Q_{\mathbf{R}}\mathbf{x} = 0. \tag{14}$$

We also have the following result:

Lemma:
Let

$$\mathbf{Bx} = \mathbf{g} \tag{15}$$

where B is given by (9), and assume that (a) g and x are 2π-periodic vector functions satisfying

$$(Q_N, \bar{Q}_N)(\mathbf{x}, \mathbf{g}) = 0. \tag{16}$$

If it is further assumed that (b) none of the eigenvalues of the problem

$$[-\lambda\mathbf{M} + \mathbf{K}]\mathbf{z} = 0,\ \mathbf{z}^T\mathbf{z} = 1, \tag{17}$$

is of the type $n^2\lambda_0$, $n = 0, 2, 3, 4, \ldots$, then there exists a unique solution of (15).

An informal proof of this Lemma is quite simple. First both $\mathbf{x}$ and $\mathbf{g}$ are expanded in Fourier series in t, i.e. we set

$$\mathbf{g} = \mathbf{g}_0 + \sum_{n=1}^{\infty} (\mathbf{g}_n\,\mathrm{e}^{int} + \bar{\mathbf{g}}_n\,\mathrm{e}^{-int});\ \mathbf{x} = \mathbf{x}_0 + \sum_{n=1}^{\infty} (\mathbf{x}_n\,\mathrm{e}^{int} + \bar{\mathbf{x}}_n\,\mathrm{e}^{-int}). \tag{18}$$

If these equations are substituted into (15), we obtain

$$[-n^2\lambda_0\mathbf{M} + \mathbf{K}]\mathbf{x}_n = \mathbf{g}_n. \tag{19}$$

Obviously, by virtue of the assumption (b) of the Lemma, eqn (19) is solvable uniquely for all $n \neq 1$. For $n = 1$, we note that because of the assumption (16), both $\mathbf{x}_1$ and $\mathbf{g}_1$ are restricted to be orthogonal to $\mathbf{y}$, which is the solution of the homogeneous problem (8a). Hence $\mathbf{x}_1$ can also be determined uniquely.

Application of the Lyapunov–Schmidt method[1] to the nonlinear oscillations problem proceeds along the same lines as in other bifurcation problems. Thus, the solution is written as

$$\mathbf{q} = (\alpha\,\mathrm{e}^{it} + \bar{\alpha}\,\mathrm{e}^{-it})\mathbf{y} + \mathbf{w};\ (Q_N, \bar{Q}_N)\mathbf{w} = 0. \tag{20a, b}$$

We now use (14) to write (4) in the equivalent form

$$\mathbf{Bw} + \mathbf{Q}_R\mathbf{d} = 0, \tag{21a}$$

$$Q_N\mathbf{d} = \bar{Q}_N\mathbf{d} = 0, \tag{21b}$$

where

$$\begin{aligned}\mathbf{d} \equiv -(\lambda - \lambda_0)\mathbf{My}(\alpha\,\mathrm{e}^{it} + \bar{\alpha}\,\mathrm{e}^{-it}) + (\lambda - \lambda_0)\mathbf{M}\ddot{\mathbf{w}} \\ + \mathbf{N}[(\alpha\,\mathrm{e}^{it} + \bar{\alpha}\,\mathrm{e}^{-it})\mathbf{y} + \mathbf{w}] \\ + \mu[\sqrt{(\lambda)}i(\alpha\,\mathrm{e}^{it} - \bar{\alpha}\,\mathrm{e}^{-it})\mathbf{Cy} + \sqrt{(\lambda)}\mathbf{C}\dot{\mathbf{w}} - \mathbf{f}(\mathrm{e}^{it} + \mathrm{e}^{-it})].\end{aligned} \tag{22}$$

We have thus reduced the original problem to two sets, one in the range and the other in the null-space of $\mathbf{B}$. (Note that if $\mathbf{d}$ is real, which is the case for problems under consideration, only one of the two equations (21b) is independent.) We first solve (21a) for $\mathbf{w}$ as a function of α, $\bar{\alpha}$, λ and μ; that this function can be obtained uniquely and is analytic is assured by the Lemma given above and the Implicit Function Theorem. A solution adequate for many applications is

$$\mathbf{w} = (\alpha^2\,\mathrm{e}^{2it} + \bar{\alpha}^2\,\mathrm{e}^{-2it})\mathbf{w}^{(2)} + \alpha\bar{\alpha}\mathbf{w}^{(0)} + \cdots \tag{23}$$

where $\mathbf{w}^{(2)}$ and $\mathbf{w}^{(0)}$ are obtained from the algebraic equations

$$[-4\lambda_0\mathbf{M}+\mathbf{K}]\mathbf{w}^{(2)}+\frac{1}{2}\mathbf{N1}(\mathbf{y})\mathbf{y}=0, \tag{24a}$$

$$\mathbf{K}\mathbf{w}^{(0)}+\mathbf{N1}(\mathbf{y})\mathbf{y}=0. \tag{24b}$$

These equations are obtained by substituting (23) into (21a) and (22) and requiring coefficients of α^2, $\bar{\alpha}^2$ and $\alpha\bar{\alpha}$ to vanish. Further, substitution of (23) into (21b) leads to the final result

$$-(\omega^2-\omega_0{}^2)m\alpha+\gamma\alpha^2\bar{\alpha}+\mu[i\omega c\alpha-\zeta]=0, \tag{25}$$

where m and c denote the usual modal mass and damping, given by

$$(m, c)=y^T(\mathbf{M},\mathbf{C})\mathbf{y}, \tag{26a}$$

and

$$\zeta=\mathbf{y}^T\mathbf{f}, \tag{26b}$$

$$\gamma=\mathbf{y}^T[\mathbf{N1}(\mathbf{y})(\mathbf{w}^{(2)}+\mathbf{w}^{(0)})+\mathbf{N2}(\mathbf{y})\mathbf{y}]. \tag{26c}$$

Equation (25) is a (complex) nonlinear relation between the (complex) amplitude of oscillation and the other parameters. It can be used to obtain both the amplitude and phase of the steady state solution of (1).

It may be pointed out here that in absence of damping and the forcing function, eqn (25) furnishes

$$-(\omega^2-\omega_0{}^2)m+\gamma\alpha\bar{\alpha}=0 \tag{27a}$$

or

$$\omega^2=\omega_0{}^2+\frac{\gamma}{4m}A^2:\ A^2\equiv 4\alpha\bar{\alpha}. \tag{27b}$$

which is the amplitude frequency equation for free-vibration of the strucuture, with A being the (real) amplitude of oscillation of the linear free vibration mode.

Equations (23)–(26) are the main results of our analysis. According to these results, if there is a quadratically nonlinear term in the equation of motion, one has to calculate the participating modes $\mathbf{w}^{(2)}$ and $\mathbf{w}^{(0)}$ from (24) in addition to the free vibration mode in order to calculate the scalar γ in (25) which determines the essential nonlinear behavior. Conversely, if only cubic terms appear in the equations of motion, only the linear free vibration mode is needed to calculate the coefficients in the amplitude frequency equation (25).

It is appropriate to end this note with some comments on a method which has been widely used for the solution of the nonlinear oscillations problems (see, e.g. [5, 6]). In our notation the technique appears to be based on the solution of the nonlinear eigenvalue problem

$$\mathbf{K}\mathbf{q}+\frac{1}{2}\mathbf{N1}(\mathbf{q})\mathbf{q}+\frac{1}{3}\mathbf{N2}(\mathbf{q})\mathbf{q}=\Lambda\mathbf{M}\mathbf{q}. \tag{28}$$

using the iterative scheme

$$\left[\mathbf{K}+\frac{1}{2}\mathbf{N1}(\mathbf{q}^{(i)})\ +\frac{1}{3}\mathbf{N2}(\mathbf{q}^{(i)})\right]\mathbf{q}^{(i+1)}=\Lambda^{(i+1)}\mathbf{M}\mathbf{q}^{(i+1)}$$

$$i=1,2,,\ldots \tag{29}$$

which defines a sequence of linear eigenvalue problems. It is not only difficult to see how (28) is derived from the basic problem (4) but, also, the application of (28) to the Duffing's equation

$$\ddot{x} + \omega_0^2 x + \alpha x^3 = 0 \tag{30}$$

leads to the amplitude frequency equation

$$\omega^2 = \omega_0^2 + \alpha A^2. \tag{31}$$

In contrast, the correct result, obtainable by using either our procedure or any of the classical perturbation methods[7], contains a factor of 3/4 before the last term in (31). It would thus appear that the method utilized in [5, 6] is not applicable to the general nonlinear oscillations problem for elastic systems.

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